

Solution of the parametric center problem for the Abel differential equation

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Abstract

The Abel differential equation $y' = p(x)y^2 + q(x)y^3$ with $p, q \in \mathbb{R}[x]$ is said to have a center on a segment $[a, b]$ if all its solutions, with the initial value $y(a)$ small enough, satisfy the condition $y(b) = y(a)$. The problem of description of conditions implying that the Abel equation has a center may be interpreted as a simplified version of the classical Center-Focus problem of Poincaré. The Abel equation is said to have a “parametric center” if for each $\varepsilon \in \mathbb{R}$ the equation $y' = p(x)y^2 + \varepsilon q(x)y^3$ has a center. In this paper we show that the Abel equation has a parametric center if and only if the antiderivatives $P = \int p(x)dx$, $Q = \int q(x)dx$ satisfy the equalities $P = \tilde{P} \circ W$, $Q = \tilde{Q} \circ W$ for some polynomials \tilde{P} , \tilde{Q} , and W such that $W(a) = W(b)$. We also show that the last condition is necessary and sufficient for the “generalized moments” $\int_a^b P^i dQ$ and $\int_a^b Q^i dP$ to vanish for all $i \geq 0$.

1 Introduction

Let

$$y' = p(x)y^2 + q(x)y^3 \quad (1)$$

be the Abel differential equation, where x is real and $p(x)$ and $q(x)$ are continuous. Equation (1) is said to have a center on a segment $[a, b]$ if all its solutions, with the initial value $y(a)$ small enough, satisfy the condition $y(b) = y(a)$.

The problem of description of conditions implying a center for (1) is closely related with the classical Poincaré center-focus problem about conditions implying that all solutions of the system

$$\begin{cases} \dot{x} &= -y + F(x, y), \\ \dot{y} &= x + G(x, y), \end{cases} \quad (2)$$

where $F(x, y)$, $G(x, y)$ are polynomials without constant and linear terms, around zero are closed. Namely, it was shown in [9] that if $F(x, y)$, $G(x, y)$ are homogeneous polynomials of the same degree, then one can construct trigonometric polynomials $f(\cos \varphi, \sin \varphi)$, $g(\cos \varphi, \sin \varphi)$ such that (2) has a center if and only if all solutions of the equation

$$\frac{dr}{d\varphi} = f(\cos \varphi, \sin \varphi) r^2 + g(\cos \varphi, \sin \varphi) r^3$$

with $r(0)$ small enough are periodic on $[0, 2\pi]$.

Set

$$P(x) = \int_0^x p(s)ds, \quad Q(x) = \int_0^x q(s)ds. \quad (3)$$

The following “composition condition” introduced in [1] is sufficient for equation (1) to have a center: there exist C^1 -functions \tilde{P}, \tilde{Q}, W such that

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)), \quad W(a) = W(b). \quad (4)$$

Indeed, if (4) holds, then any solution of (1) has the form $y(x) = \tilde{y}(W(x))$, where \tilde{y} is a solution of the equation

$$y' = \tilde{P}'(x)y^3 + \tilde{Q}'(x)y^2,$$

implying that $y(a) = y(b)$, since $W(a) = W(b)$.

It is known that in general the composition condition is not necessary for (1) to have a center ([2]). However, it is believed that, in the case where $p(x)$ and $q(x)$ are polynomials, equation (1) has a center if and only if the composition condition (4) holds for some polynomials $\tilde{P}, \tilde{Q}, W \in \mathbb{R}[x]$ (see [6], [8] for some partial results in this direction).

In this paper we study the following “parametric center problem” for equation (1) with polynomial coefficients: *under what conditions the equation*

$$y' = p(x)y^2 + \varepsilon q(x)y^3, \quad p, q \in \mathbb{R}[x], \quad (5)$$

has a center for any $\varepsilon \in \mathbb{R}$? Posed for the first time in the series of papers [3], [4], [5], this problem turned out to be very constructive and resulted in a whole area of new ideas and methods related to the so called “polynomial moment problem” (see the discussion below). However, in its full generality the parametric center problem remained unsolved (see the recent paper [7] for the state of the art), and the goal of this paper is to fill this gap. Our main result is the following theorem.

Theorem 1.1 *Abel differential equation (5) has a center on a segment $[a, b]$ for any $\varepsilon \in \mathbb{R}$ if and only if the antiderivatives $P = \int p(x)dx$ and $Q = \int q(x)dx$ satisfy composition condition (4) for some polynomials \tilde{P}, \tilde{Q}, W .*

The proof of Theorem 1.1 is based on the link between the parametric center problem and the “polynomial moment problem”. Namely, it was shown in [5] that the parametric center implies the equalities

$$\int_a^b P^i dQ = 0, \quad i \geq 0, \quad \int_a^b Q^i dP = 0, \quad i \geq 0. \quad (6)$$

We will call the problem of description of solutions (6) the “mixed polynomial moment problem”, and the problem of description of solutions

$$\int_a^b P^i dQ = 0, \quad i \geq 0, \quad (7)$$

simply the “polynomial moment problem”.

The polynomial moment problem has been studied in many recent papers (see e.g. [3], [4], [5], [10], [11], [12], [13], [14], [15], [16], [17], [19]). Again, the composition condition (4) is sufficient for equalities (7) to be satisfied although in general is not necessary ([11]). A complete solution of the polynomial moment problem was obtained in the recent papers [15], [17]. Namely, it was shown in [15] that if polynomials P, Q satisfy (7), then there exist polynomials Q_j such that $Q = \sum_j Q_j$ and

$$P(x) = P_j(W_j(x)), \quad Q_j(x) = V_j(W_j(x)), \quad W_j(a) = W_j(b) \quad (8)$$

for some polynomials $P_j(z), V_j(z), W_j(z)$. Moreover, in [17] polynomial solutions of (7) were described in an explicit form (see Section 2 below).

In this paper we apply results of [17] to each of the two systems appeared in (6) separately and show that the restrictions obtained imply that any solution P, Q of the mixed polynomial moment problem satisfy composition condition (4). Thus, in fact we prove the following “moment” counterpart of Theorem 1.1.

Theorem 1.2 *Polynomials $P, Q \in \mathbb{R}[x]$ satisfy equalities (6) if and only if they satisfy composition condition (4) for some polynomials $\tilde{P}, \tilde{Q}, W \in \mathbb{R}[x]$.*

Although the center problem for the Abel equation with polynomial coefficients can be considered in the complex setting, in this paper we work in the classical real framework. Thus, we start the paper from the adaptation for the real case of the results of [17], obtained over \mathbb{C} . Namely, we show in Section 2 that possible “types” of solutions of the polynomial moment problem over \mathbb{R} remain the same, although one of these types becomes “smaller” (Theorem 2.10). Besides, in Section 2 we establish some important restrictions of the arithmetical nature on points a, b for which there exist solutions of (7) which do not satisfy the composition condition (Corollary 2.5).

In Section 3 we apply the results of Section 2 to (6), and prove Theorem 1.2. The main difficulties of the proof stem from the fact that after solving systems in (6) separately we arrive to functional equations of the type

$$\sum_{j=1}^r V_j(W_j(x)) = A(B(z)), \quad (9)$$

where V_j, W_j, A, B are polynomials, and r equals 2 or 3. Such equations can be considered as generalizations of the functional equation

$$A(B(x)) = C(D(x)), \quad (10)$$

studied by Ritt ([18]). However, the well established methods for studying (10), related to the monodromy, cannot be applied to (9) for $r > 1$, and essentially the only method which remains is a painstaking analysis of coefficients. Such an analysis in general leads to rather complicated systems of equations, and Theorem 1.2 is deduced from restrictions on P and Q obtained from these systems combined with restrictions on possible values of a and b .

2 Polynomial moment problem over \mathbb{C} and over \mathbb{R}

2.1 Solution of the polynomial moment problem over \mathbb{C}

In this subsection we briefly recall a description of $P, Q \in \mathbb{C}[z]$ satisfying (7) for $a, b \in \mathbb{C}$. For more details we refer the reader to [17].

Recall that the Chebyshev polynomials (of the first kind) T_n are defined by the formula $T_n(\cos \varphi) = \cos(n\varphi)$. It follows directly from this definition that

$$T_n(1) = 1, \quad T_n(-1) = (-1)^n, \quad n \geq 0 \quad (11)$$

and

$$T_n \circ T_m = T_m \circ T_n = T_{mn}, \quad n, m \geq 1,$$

where the symbol \circ denotes a composition of functions, $A \circ B = A(B(z))$.

An explicit expression for T_n is given by the formula

$$T_n = \frac{n}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \quad (12)$$

implying in particular that

$$T_n(-x) = (-1)^n T_n(x). \quad (13)$$

Following [17], we will call a solution P, Q of (7) reducible if composition condition (4) holds for some $\tilde{P}, \tilde{Q}, W \in \mathbb{C}[x]$.

Theorem 2.1 ([17]) *Let P, Q be non-constant complex polynomials and a, b distinct complex numbers such that equalities (7) hold. Then, either Q is a reducible solution of (7), or there exist complex polynomials $P_j, Q_j, V_j, W_j, 1 \leq j \leq r$, such that*

$$Q = \sum_{j=1}^r Q_j, \quad P = P_j \circ W_j, \quad Q_j = V_j \circ W_j, \quad W_j(a) = W_j(b).$$

Moreover, one of the following conditions holds:

1) $r = 2$ and

$$P = U \circ z^{sn} R^n(z^n) \circ V, \quad W_1 = z^n \circ V, \quad W_2 = z^s R(z^n) \circ V,$$

where R, U, V are complex polynomials, $n > 1, s > 0, \text{GCD}(s, n) = 1$;

2) $r = 2$ and

$$P = U \circ T_{m_1 m_2} \circ V, \quad W_1 = T_{m_1} \circ V, \quad W_2 = T_{m_2} \circ V,$$

where U, V are complex polynomials, $m_1 > 1, m_2 > 1, \text{GCD}(m_1, m_2) = 1$;

3) $r = 3$ and

$$P = U \circ z^2 R^2(z^2) \circ T_{m_1 m_2} \circ V,$$

$$W_1 = T_{2m_1} \circ V, \quad W_2 = T_{2m_2} \circ V, \quad W_3 = (zR(z^2) \circ T_{m_1 m_2}) \circ V,$$

where R, U, V are complex polynomials, $m_1 > 1$, $m_2 > 1$ are odd, and $\text{GCD}(m_1, m_2) = 1$.

We will call solutions 1), 2), 3) appearing in Theorem 2.1 solutions of the first, the second, and the third type correspondingly.

Notice that these sets of solutions are not disjointed. For example, if one of the parameters n, m of a solution of the second type equals 2, then this solution is also a solution of the first type. Indeed, if say $n = 2$, then $W_1 = T_2 \circ V = \mu \circ z^2 \circ V$, where $\mu = 2x - 1$. On the other hand, since m is odd in view of $\text{GCD}(n, m) = 1$, the polynomial $W_2 = T_m \circ V$ has the form $T_m = zR(z^2) \circ V$ by (12). Therefore,

$$P = (U \circ \mu) \circ z^2 R^2(z^2) \circ V, \quad Q = ((V_1 \circ \mu) \circ z^2 + V_2 \circ zR(z^2)) \circ V,$$

and for the polynomials $\widetilde{W}_1 = z^2 \circ V$ and $\widetilde{W}_2 = W_2 = zR(z^2) \circ V$ the equalities

$$\widetilde{W}_1(a) = \widetilde{W}_1(b), \quad \widetilde{W}_2(a) = \widetilde{W}_2(b) \quad (14)$$

hold.

Similarly, if the parameters a, b of a solution of the third type satisfy $V(a) = -V(b)$, then this solution is also a solution of the first type. Indeed,

$$V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} = \widetilde{V}_1 \circ z^2$$

for some $\widetilde{V}_1 \in \mathbb{C}[z]$, while

$$zR(z^2) \circ T_{m_1 m_2} = z\widetilde{R}(z^2)$$

for some $\widetilde{R} \in \mathbb{C}[z]$, since m_1, m_2 are odd. Therefore,

$$P = U \circ z^2 \widetilde{R}^2(z^2) \circ V, \quad Q = (\widetilde{V}_1 \circ z^2 + V_3 \circ z\widetilde{R}(z^2)) \circ V,$$

and $\widetilde{W}_1 = z^2 \circ V$ satisfies $\widetilde{W}_1(a) = \widetilde{W}_1(b)$, since $V(a) = -V(b)$.

Finally, one can check that a solution of the third type is a solution of the second type if $(T_{m_i} \circ V)(a) = (T_{m_i} \circ V)(b)$ for i equals 1 or 2 (for more details concerning interrelations between different types of solutions see [17], pp. 725-726).

2.2 Lemmas related to a, b

In this subsection we collect some results implying restrictions on the integration limits a, b appearing in solutions of the second and the third types.

Lemma 2.2 *Let $T_{m_1}, T_{m_2}, T_{m_3}$ be the Chebyshev polynomials and a, b be distinct complex numbers.*

a) *Assume that*

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b). \quad (15)$$

Then either $T_l(a) = T_l(b)$ for $l = \text{GCD}(m_1, m_2)$, or

$$T'_{m_1 m_2}(a) = T'_{m_1 m_2}(b) = 0. \quad (16)$$

b) *Assume that*

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b), \quad T_{m_3}(a) = T_{m_3}(b). \quad (17)$$

Then there exists a pair of distinct indices i_1, i_2 , $1 \leq i_1, i_2 \leq 3$, such that $T_l(a) = T_l(b)$ for $l = \text{GCD}(m_{i_1}, m_{i_2})$.

Proof. Choose $\alpha, \beta \in \mathbb{C}$ such that $\cos \alpha = a$, $\cos \beta = b$. Then equalities (15) imply the equalities

$$m_1 \alpha = \varepsilon_1 m_1 \beta + 2\pi k_1, \quad m_2 \alpha = \varepsilon_2 m_2 \beta + 2\pi k_2, \quad (18)$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, and $k_1, k_2 \in \mathbb{Z}$. Assume first that $\varepsilon_1 = \varepsilon_2$. Let u, v be integers satisfying

$$um_1 + vm_2 = l. \quad (19)$$

Multiplying the first equality in (18) by u and adding the second equality multiplied by v , we see that

$$l\alpha = \varepsilon_1 l\beta + 2\pi k_1 u + 2\pi k_2 v,$$

implying that $T_l(a) = T_l(b)$.

Assume now that $\varepsilon_1 = -\varepsilon_2$. Then, similarly, we conclude that

$$l\alpha = \varepsilon_1 \beta (um_1 - vm_2) + 2\pi k_1 u + 2\pi k_2 v. \quad (20)$$

Furthermore, eliminating α from equalities (18) we obtain

$$\varepsilon_1 m_1 m_2 \beta = \pi k_2 m_1 - \pi k_1 m_2. \quad (21)$$

Since

$$T'_n(\cos \varphi) = n(\sin n\varphi / \sin \varphi), \quad (22)$$

equality (21) implies that $T'_{m_1 m_2}(b) = 0$, unless $\beta = \pi k_3$, $k_3 \in \mathbb{Z}$. In the last case, $b = 1$ if k_3 is even, and $b = -1$ if k_3 is odd, implying that

$$T_l(b) = (-1)^{k_3 l},$$

in view of (11). On the other hand, if $\beta = \pi k_3$, then (20) implies that

$$T_l(a) = (-1)^{k_3(um_1 - vm_2)}.$$

Since the sum and the difference of any two numbers have the same parity this implies that

$$T_l(a) = (-1)^{k_3(um_1 + vm_2)} = (-1)^{k_3 l} = T_l(b).$$

Similarly, one can see that $T_l(a) = T_l(b)$ unless $T'_{m_1 m_2}(a) = 0$.

In order to prove b) observe that equalities (17) imply the equalities

$$m_1 \alpha = \varepsilon_1 m_1 \beta + 2\pi k_1, \quad m_2 \alpha = \varepsilon_2 m_2 \beta + 2\pi k_2, \quad m_3 \alpha = \varepsilon_3 m_3 \beta + 2\pi k_3, \quad (23)$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, $\varepsilon_3 = \pm 1$, and $k_1, k_2, k_3 \in \mathbb{Z}$. Clearly, among the numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ at least two are equal and we conclude as above that the equality $T_l(a) = T_l(b)$ holds for $l = \text{GCD}(m_{i_1}, m_{i_2})$, where $\varepsilon_{i_1} = \varepsilon_{i_2}$. \square

Corollary 2.3 *Let T_{m_1}, T_{m_2} be the Chebyshev polynomials and a, b be distinct complex numbers. Assume that*

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b)$$

and $\text{GCD}(m_1, m_2) = 1$. Then

$$T'_{m_1 m_2}(a) = T'_{m_1 m_2}(b) = 0.$$

Proof. Follows from Lemma 2.2, a) taking into account that the equality $S(a) = S(b)$ for some polynomial S and $a \neq b$ obviously implies that $\deg S > 1$. \square

Recall that a number $\gamma \in \mathbb{C}$ is called algebraic if it is a root of an equation with rational coefficients. The set of all algebraic numbers is a subfield of \mathbb{C} . A monic polynomial $p(x) \in \mathbb{Q}[x]$ of the minimal degree such that $p(\gamma) = 0$ is called a minimal polynomial of γ . A minimal polynomial is irreducible over \mathbb{Q} . An algebraic number γ is called an algebraic integer if its minimal polynomial has integer coefficients. In fact, this condition may be replaced by a weaker condition that γ is a root of *some* monic polynomial with integer coefficients. The set of all algebraic integers is closed under addition and multiplication.

Lemma 2.4 Assume that $a \in \mathbb{C}$ is a root of T'_n . Then $a \in \mathbb{R}$, and $2a$ is an algebraic integer.

Proof. Since equality (22) shows that T'_n has $n - 1$ distinct real roots, all roots of T'_n are real. The other statements follow from the formulas

$$T'_n = nU_{n-1}$$

and

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k},$$

where U_n denotes the Chebyshev polynomial of the second kind. \square

Corollary 2.5 In the notation of Theorem 2.1 assume that Q is a solution of (7) of the second type, or a solution of the third type, which cannot be represented as a solution of the first type. Then $2V(a)$ and $2V(b)$ are algebraic integers.

Proof. Without loss of generality we may assume that $V = x$. If Q is of the second type, then the statement follows from Corollary 2.3 and Lemma 2.4.

If Q is of the third type, then applying Lemma 2.2, a) and Lemma 2.4 to the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \quad T_{2m_2}(a) = T_{2m_2}(b),$$

we conclude that $2V(a)$ and $2V(b)$ are algebraic integers, unless $T_2(a) = T_2(b)$. However, the last equality yields the equality $a = -b$, implying, as it was observed above, that Q can be represented as a solution of the first type. \square

2.3 Decompositions of polynomials with real coefficients

In this subsection we collect necessary results concerning decomposition of polynomials with real coefficients into compositions of polynomials of lesser degree.

The following lemma is well known (see e. g. Corollary 2.2 in [17]).

Lemma 2.6 Assume that

$$P = A \circ B = \tilde{A} \circ \tilde{B},$$

where $P, A, B, \tilde{A}, \tilde{B} \in \mathbb{C}[z]$ and $\deg A = \deg \tilde{A}$. Then there exists a polynomial $\mu \in \mathbb{C}[z]$ of degree one such that

$$\tilde{A} = A \circ \mu^{-1}, \quad \tilde{B} = \mu \circ B. \quad \square$$

Corollary 2.7 Let $P = U \circ V$, where $P \in \mathbb{R}[x]$, while $U, V \in \mathbb{C}[z]$. Assume that the leading coefficient of V and its constant term are real numbers. Then $U, V \in \mathbb{R}[x]$.

Proof. Since $P \in \mathbb{R}[x]$, we have:

$$P = U \circ V = \overline{U} \circ \overline{V}, \quad (24)$$

where $\overline{U}, \overline{V}$ are polynomials obtained from U, V by the complex conjugation of all coefficients. By Lemma 2.6, equality (24) implies that

$$\overline{U} = U \circ \mu^{-1}, \quad \overline{V} = \mu \circ V, \quad (25)$$

where $\mu = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$. Since the leading coefficient of V is real, the second equality in (25) implies that $\alpha = 1$. Now the equality $\overline{V} = V + \beta$ implies that $\beta = 0$, since the constant term of V is real. Therefore, $\overline{U} = U$, $\overline{V} = V$ and hence $U, V \in \mathbb{R}[x]$. \square

Corollary 2.8 Assume that

$$P = U \circ V,$$

where $P \in \mathbb{R}[x]$, while $U, V \in \mathbb{C}[z]$. Then there exists a polynomial $\mu \in \mathbb{C}[z]$ of degree one such that the polynomials

$$U_1 = U \circ \mu^{-1}, \quad V_1 = \mu \circ V$$

are contained in $\mathbb{R}[x]$.

Proof. Let μ be any polynomial of degree one such that the leading coefficient and the constant term of the polynomial $V_1 = \mu \circ V$ are real numbers. Then U_1 and V_1 are contained in $\mathbb{R}[x]$ by Corollary 2.7. \square

Lemma 2.9 *Let μ_1, μ_2 be complex polynomials of degree one.*

a) *Assume that the polynomial $\mu_1 \circ z^n \circ \mu_2$, $n \geq 2$, has real coefficients. Then there exist $\tilde{\mu} \in \mathbb{R}[x]$ and $c \in \mathbb{C}$ such that $\mu_2 = c\tilde{\mu}$.*

b) *Assume that the polynomial $\mu_1 \circ T_n \circ \mu_2$, $n \geq 2$, has real coefficients. Then either $\mu_2 \in \mathbb{R}[x]$ or there exists $\tilde{\mu} \in \mathbb{R}[x]$ such that $\mu_2 = i\tilde{\mu}$.*

Proof. Let $\mu_1 = \alpha_1 z + \beta_1$, $\mu_2 = \alpha_2 z + \beta_2$, where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$. Then the coefficients of z^n and z^{n-1} of the polynomial $\mu_1 \circ z^n \circ \mu_2$ are $c_n = \alpha_1 \alpha_2^n$ and $c_{n-1} = \alpha_1 \alpha_2^{n-1} \beta_2 n$, correspondingly. Since by assumption these numbers are real, we conclude that

$$\frac{c_{n-1}}{c_n} = \frac{n\beta_2}{\alpha_2}$$

is also a real number. Therefore, $\beta_2/\alpha_2 \in \mathbb{R}$, and hence $\mu_2 = \alpha_2 \tilde{\mu}$, where $\tilde{\mu} = x + (\beta_2/\alpha_2) \in \mathbb{R}[x]$.

Similarly, since

$$T_n = 2^{n-1}x^n - n2^{n-3}x^{n-2} + \dots$$

by (12), the coefficients of z^n , z^{n-1} , z^{n-2} of the polynomial $\mu_1 \circ T_n \circ \mu_2$ are:

$$c_n = \alpha_1 2^{n-1} \alpha_2^n, \quad c_{n-1} = \alpha_1 2^{n-1} n \alpha_2^{n-1} \beta_2, \quad c_{n-2} = \alpha_1 2^{n-2} n(n-1) \alpha_2^{n-2} \beta_2^2 - \alpha_1 2^{n-3} n \alpha_2^{n-2},$$

correspondingly. As above, $c_n, c_{n-1} \in \mathbb{R}$ implies that $\beta_2/\alpha_2 \in \mathbb{R}$. Since

$$c_{n-2} = \frac{n(n-1)c_n}{2} \left(\frac{\beta_2}{\alpha_2} \right)^2 - \frac{nc_n}{4} \left(\frac{1}{\alpha_2} \right)^2,$$

it follows now from $c_{n-2} \in \mathbb{R}$ that $\alpha_2^2 \in \mathbb{R}$ implying the statement. \square

2.4 Solution of the polynomial moment problem over \mathbb{R}

In this subsection we deduce from Theorem 2.1 a description of polynomials P, Q with *real* coefficients satisfying (7) for $a, b \in \mathbb{R}$.

The theorem below is a “real” analogue of Theorem 2.1. Keeping the above notation we will call a solution $Q \in \mathbb{R}[x]$ of (7) reducible if (4) holds for some $\tilde{P}, \tilde{Q}, W \in \mathbb{R}[x]$. We also will call solutions 1), 2), 3) described below solutions of the first, the second, and the third type. Notice that the set of solutions of the first type in the real case is “smaller” than the one in the complex case.

Theorem 2.10 *Let P, Q be non-constant real polynomials and a, b distinct real numbers such that equalities (7) hold. Then, either Q is a reducible solution of (7), or there exist real polynomials P_j, Q_j, V_j, W_j , $1 \leq j \leq r$, such that*

$$Q = \sum_{j=1}^r Q_j, \quad P = P_j \circ W_j, \quad Q_j = V_j \circ W_j, \quad W_j(a) = W_j(b).$$

Moreover, one of the following conditions holds:

1) $r = 2$ and

$$P = U \circ x^2 R^2(x^2) \circ V, \quad W_1 = x^2 \circ V, \quad W_2 = xR(x^2) \circ V,$$

where R, U, V are real polynomials;

2) $r = 2$ and

$$P = U \circ T_{m_1 m_2} \circ V, \quad W_1 = T_{m_1} \circ V, \quad W_2 = T_{m_2} \circ V,$$

where U, V are real polynomials, $m_1 > 1$, $m_2 > 1$, $\text{GCD}(m_1, m_2) = 1$;

3) $r = 3$ and

$$P = U \circ x^2 R^2(x^2) \circ T_{m_1 m_2} \circ V,$$

$$W_1 = T_{2m_1} \circ V, \quad W_2 = T_{2m_2} \circ V, \quad W_3 = (xR(x^2) \circ T_{m_1 m_2}) \circ V,$$

where R, U, V are real polynomials, $m_1 > 1, m_2 > 1$ are odd, and $\text{GCD}(m_1, m_2) = 1$.

Proof. Our strategy is to apply Theorem 2.1 and to use the condition that $P, Q \in \mathbb{R}[x]$ and $a, b \in \mathbb{R}$. Assume first that (4) holds for some $\tilde{P}, \tilde{Q}, W \in \mathbb{C}[z]$. Applying Corollary 2.8 to the equality $P = \tilde{P} \circ W$ we conclude that without loss of generality we may assume that \tilde{P} and W are contained in $\mathbb{R}[x]$. Now the equality $Q = \tilde{Q} \circ W$ implies by Corollary 2.7 that \tilde{Q} also is contained in $\mathbb{R}[x]$.

Assume that Q is a solution of the first type. It follows from the equality $P = P_1 \circ W_1$ by Corollary 2.8 that there exists a complex polynomial of degree one μ_1 such that the polynomial $\mu_1 \circ W_1$ has real coefficients. Further, applying Corollary 2.8 to the equality $\mu_1 \circ W_1 = \mu_1 \circ z^n \circ V$, we conclude that there exists a complex polynomial of degree one μ_2 such that the polynomials $\mu_1 \circ z^n \circ \mu_2$ and $\mu_2^{-1} \circ V$ have real coefficients. By Lemma 2.9, a) this implies that there exist $\tilde{\mu} \in \mathbb{R}[x]$ and $c \in \mathbb{C}$ such that $\mu_2 = c\tilde{\mu}$. Since $\mu_2^{-1} = \tilde{\mu}^{-1} \circ z/c$, it follows now from $\mu_2^{-1} \circ V \in \mathbb{R}[x]$ that $V/c \in \mathbb{R}[x]$. Therefore, changing the polynomial V to V/c , and modifying the polynomials P_1, V_1 , and R in an obvious way, without loss of generality we may assume that $V \in \mathbb{R}[x]$.

Clearly, $V \in \mathbb{R}[x]$ implies that $W_1 = z^n \circ V \in \mathbb{R}[x]$. It follows now from $P = P_1 \circ W_1$ by Corollary 2.7 that $P_1 \in \mathbb{R}[x]$. Furthermore, it follows from $W_1(a) = W_1(b)$ and $a, b \in \mathbb{R}$ that $n = 2k$, and $V(a) = -V(b)$. Since $\text{GCD}(s, n) = 1$, the polynomial $z^s R(z^n)$ has the form $z\tilde{R}(z^2)$ for some $\tilde{R} \in \mathbb{C}[z]$. Thus, changing P_1 to $P_1 \circ z^{n/2}$ and $z^s R(z^n)$ to $z\tilde{R}(z^2)$, without loss of generality we may assume that $W_1 = z^2 \circ V$ and $W_2 = zR(z^2) \circ V$. Applying Corollary 2.7 to the equality $P = (P_2 \circ zR(z^2)) \circ V$ we see that $P_2 \circ zR(z^2) \in \mathbb{R}[x]$. Therefore, taking into account that the constant term of $zR(z^2)$ is zero, Corollary 2.7 implies that for $c \in \mathbb{C}$ such that the leading coefficient of $czR(z^2)$ is real the polynomials $P_2 \circ z/c$ and $czR(z^2)$ are contained in $\mathbb{R}[x]$. Thus, modifying the polynomials P_2 and R we can assume that they are contained in $\mathbb{R}[x]$. Now Corollary 2.7 applied to the equality $P = U \circ (z^2 R^2(z^2) \circ V)$ implies that $U \in \mathbb{R}[x]$.

Finally, the equality

$$Q = V_1 \circ W_1 + V_2 \circ W_2 = \overline{V}_1 \circ W_1 + \overline{V}_2 \circ W_2$$

implies that

$$Q = \frac{V_1 + \overline{V}_1}{2} \circ W_1 + \frac{V_2 + \overline{V}_2}{2} \circ W_2.$$

Therefore, changing if necessary V_1 to $(V_1 + \overline{V}_1)/2$ and V_2 to $(V_2 + \overline{V}_2)/2$, without loss of generality we may assume that $V_1, V_2 \in \mathbb{R}[x]$.

Assume now that Q is a solution of the third type. We may assume that $V(a) \neq -V(b)$, for otherwise, as it was observed after Theorem 2.1, this solution also belongs to the first type considered earlier. As above, we conclude that there exist complex polynomials of degree one μ_1 and μ_2 such that the polynomials $\mu_1 \circ T_{2m_1} \circ \mu_2$ and $\mu_2^{-1} \circ V$ have real coefficients. By Lemma 2.9, b), this implies that either $\mu_2 \in \mathbb{R}[x]$ or there exists $\tilde{\mu} \in \mathbb{R}[x]$ such that $\mu_2 = i\tilde{\mu}$. Since, $\mu_2^{-1} \circ V \in \mathbb{R}[x]$, in the first case $V \in \mathbb{R}[x]$, while in the second one,

$$V = i\tilde{V}, \quad \tilde{V} \in \mathbb{R}[x]. \quad (26)$$

Let us show that equality (26) is impossible. Indeed, applying Lemma 2.2, a) to the equalities $W_1(a) = W_1(b)$, $W_2(a) = W_2(b)$ and arguing as in Lemma 2.5, we conclude that the numbers $V(a)$ and $V(b)$ are roots of the polynomial $T'_{4m_1 m_2}$, for otherwise $V(a) = -V(b)$. Since T'_n has only real zeroes, we conclude that $V(a), V(b) \in \mathbb{R}$, and hence (26) is impossible in view of $a, b \in \mathbb{R}$. Thus, $V \in \mathbb{R}[x]$.

Applying now Corollary 2.7 to the equality $P = (P_3 \circ zR(z^2)) \circ (T_{m_1 m_2} \circ V)$ we conclude that $P_3 \circ zR(z^2) \in \mathbb{R}[x]$. Furthermore, arguing as above, we conclude that without loss of generality we may assume that $P_3, R \in \mathbb{R}[x]$ as well as $P_1, P_2, U \in \mathbb{R}[x]$ and $V_1, V_2, V_3 \in \mathbb{R}[x]$.

The proof of the theorem in the case where Q is a solution of the second type is obtained similarly with obvious simplifications. \square

3 Proof of Theorem 1.2

3.1 Plan of the proof

In the rest of the paper we always will assume that all considered polynomials have real coefficients.

Let us describe a general plan of the proof of Theorem 1.2. First, observe that we may assume that

$$\mathbb{R}(P, Q) = \mathbb{R}(x). \quad (27)$$

Indeed, otherwise by the Lüroth theorem, $\mathbb{R}(P, Q) = \mathbb{R}(W)$ for some $W \in \mathbb{R}(x)$, $\deg W \geq 2$, implying that

$$P = \tilde{P} \circ W, \quad Q = \tilde{Q} \circ W$$

for some $\tilde{P}, \tilde{Q} \in \mathbb{R}(x)$ such that $\mathbb{R}(\tilde{P}, \tilde{Q}) = \mathbb{R}(x)$. Moreover, since $P, Q \in \mathbb{R}[x]$, it is easy to see that we may assume that $\tilde{P}, \tilde{Q}, W \in \mathbb{R}[x]$. Therefore, since equalities (6) imply the equalities

$$\int_{W(a)}^{W(b)} \tilde{P}^i d\tilde{Q} = 0, \quad \int_{W(a)}^{W(b)} \tilde{Q}^j d\tilde{P} = 0, \quad i, j \geq 0, \quad (28)$$

it is enough to prove that if (27) holds, then P, Q cannot satisfy (6) for $a \neq b$.

Applying Theorem 2.10 to the first and to the second system of equations in (6) separately, we arrive to nine different “cases” depending on types of solutions appearing in Theorem 2.10. For example, “the case (2,1)” means that Q is a solution of the second type of the polynomial moment problem (7), while P is a solution of the first type of the polynomial moment problem

$$\int_a^b Q^j dP = 0, \quad j \geq 0.$$

In more details, this means that, from one hand,

$$Q = V_1 \circ W_1 + V_2 \circ W_2, \quad P = U \circ T_{nm} \circ V,$$

where

$$W_1 = T_n \circ V, \quad W_2 = T_m \circ V, \quad (29)$$

and

$$W_1(a) = W_1(b), \quad W_2(a) = W_2(b),$$

while, from the other hand,

$$P = \tilde{V}_1 \circ \tilde{W}_1 + \tilde{V}_2 \circ \tilde{W}_2, \quad Q = \tilde{U} \circ x^2 R^2(x^2) \circ \tilde{V},$$

where

$$\tilde{W}_1 = x^2 \circ \tilde{V}, \quad \tilde{W}_2 = xR(x^2) \circ \tilde{V},$$

and

$$\tilde{W}_1(a) = \tilde{W}_1(b), \quad \tilde{W}_2(a) = \tilde{W}_2(b).$$

In view of assumption (27), the polynomial V (as well as the polynomial \tilde{V}) is of degree one for otherwise

$$\mathbb{R}(P, Q) \subseteq \mathbb{R}(V) \subsetneq \mathbb{R}(x).$$

Furthermore, it is clear that without loss of generality we may assume that one of the polynomials V and \tilde{V} equals x . Our strategy will be to show that such systems of equations always imply that equalities (4) hold, in contradiction with (27) (recall that the condition $W(a) = W(b)$ implies that $\deg W > 1$).

Since we may exchange P and Q , it is necessary to consider only the cases (1,1), (2,1), (2,2), (3,1), (3,2), and (3,3). Finally, we may impose some additional restrictions related to the fact that a solution of the (usual) polynomial moment problem may belong to different types. For example, assuming that the theorem is already proved in the case (1,1), considering the case (2,1) we may assume that $n > 2$, $m > 2$ in (29), since otherwise the solution P, Q also belongs to the case (1,1).

For a polynomial

$$P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_i \in \mathbb{R}, \quad 0 \leq i \leq n,$$

of degree n , set

$$C_i(P) = a_{n-i}, \quad 0 \leq i \leq n.$$

The following simple lemma permits to control initial terms in a composition of two polynomials and is widely used in the following.

Lemma 3.1 *Let T be a polynomial of degree d . Then for any polynomial S of degree r with the leading coefficient c the equalities*

$$C_i(S \circ T) = C_i(cx^r \circ T), \quad 0 \leq i \leq d-1, \quad (30)$$

hold. In particular, for any two polynomials S_1, S_2 of equal degree with equal leading coefficients the equalities

$$C_i(S_1 \circ T) = C_i(S_2 \circ T), \quad 0 \leq i \leq d-1,$$

hold.

Proof. Indeed, $\deg(S - cx^r) \circ T = dr - d$. Therefore, (30) holds. \square

Corollary 3.2 *Let $T(z)$ be a polynomial of degree $d \geq 2$ such that $C_1(T) = 0$ holds, U be an arbitrary polynomial, and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. Then equality $C_1(U \circ T \circ (\alpha x + \beta)) = 0$ holds if and only if $\beta = 0$.*

Proof. Indeed, if $\deg U = r$, $C_0(U) = c$, then $C_1(U \circ T) = C_1(cx^r \circ T)$ by Lemma 3.1. On the other hand, $C_1(cx^r \circ T) = 0$, since $C_1(T) = 0$. Therefore, $C_1(U \circ T) = 0$, and it is easy to see that for any polynomial F such that $C_1(F) = 0$, the equality $C_1(F \circ (\alpha x + \beta)) = 0$ holds if and only if $\beta = 0$. \square

3.2 Proof of Theorem 1.2 in the cases (1,1)

Lemma 3.3 *Let W_1, W_2 be polynomials of degree two such that $W_1(a) = W_1(b)$, $W_2(a) = W_2(b)$ for distinct $a, b \in \mathbb{R}$. Then $W_2 = \lambda_1 W_1 + \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.*

Proof. Let

$$P = \alpha_1 x^2 + \beta_1 x + \gamma_1, \quad Q = \alpha_2 x^2 + \beta_2 x + \gamma_2,$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{R}$. Then conditions of the lemma yield the equalities

$$\alpha_1(a+b) + \beta_1 = 0, \quad \alpha_2(a+b) + \beta_2 = 0,$$

implying the statement. \square

In order to proof the theorem in the case (1,1) it is enough to observe that in this case there exist $U, R, \tilde{U}, \tilde{R} \in \mathbb{R}[x]$ such that

$$P = (U \circ zR(z^2)) \circ W_1, \quad Q = (\tilde{U} \circ z\tilde{R}(z^2)) \circ \tilde{W}_1$$

where

$$W_1 = z^2 \circ V, \quad \tilde{W}_1 = z^2 \circ \tilde{V},$$

are polynomials of degree two such that $W_1(a) = W_1(b)$, $\tilde{W}_1(a) = \tilde{W}_1(b)$. By Lemma 3.3, we have $\tilde{W}_2 = \lambda_1 W_1 + \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$, and hence (4) holds for $W = W_1$.

3.3 Proof of Theorem 1.2 in the case (2,1)

If P, Q is a solution of (6) corresponding to the case (2,1), then without loss of generality we may assume that there exist polynomials $V_1, V_2, U, \tilde{V}_1, \tilde{V}_2, \tilde{U}, R$ and $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$, such that

$$Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2} = \tilde{U} \circ x^2 R^2(x^2) \circ (\alpha x + \beta), \quad (31)$$

and

$$P = \tilde{V}_1 \circ x^2 \circ (\alpha x + \beta) + \tilde{V}_2 \circ x R(x^2) \circ (\alpha x + \beta) = U \circ T_{m_1 m_2},$$

where $\text{GCD}(m_1, m_2) = 1$. In addition, for the polynomials

$$W_1 = T_{m_1}, \quad W_2 = T_{m_2}, \quad \tilde{W}_1 = z^2 \circ (\alpha x + \beta), \quad \tilde{W}_2 = x R(x^2) \circ (\alpha x + \beta)$$

the equalities

$$W_1(a) = W_1(b), \quad W_2(a) = W_2(b), \quad \tilde{W}_1(a) = \tilde{W}_1(b), \quad \tilde{W}_2(a) = \tilde{W}_2(b) \quad (32)$$

hold. If at least one of the numbers m_1, m_2 equals 2, then P, Q belongs to the type (1,1) considered above. So, we may assume that $m_1 \geq 3, m_2 \geq 3$. Notice that the second representation for Q in (31) implies that $n = \deg Q$ is even and the inequality $n \geq 6$ holds.

Although the above conditions seem to be very strong, it is difficult to use them in their full generality since they contain many unknown parameters. Thus, actually we mostly will use only the fact that the right part of (31) is a polynomial in $x^2 \circ (\alpha x + \beta)$ together with first three equalities in (32).

First of all observe that any polynomial of the form $Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2}$ can be represented in the form

$$Q = d_n T_n + d_{n-1} T_{n-1} + \dots + d_1 T_1 + d_0, \quad d_i \in \mathbb{R}, \quad (33)$$

where $d_i = 0$, unless i is divisible either by m_1 or by m_2 . Indeed, it is clear that T_0, T_1, \dots, T_r is a basis of a subspace of $\mathbb{R}[x]$ consisting of all polynomials of degree $\leq r$. Therefore, a polynomial P can be represented in the form

$$P = V \circ T_m = (a_r x^r + a_{r-1} x^{r-1} + \dots + a_1 x + a_0) \circ T_m$$

if and only if

$$P = (b_r T_r + b_{r-1} T_{r-1} + \dots + b_1 T_1 + b_0) \circ T_m = b_r T_{rm} + b_{r-1} T_{(r-1)m} + \dots + b_1 T_m + b_0.$$

Consequently, $Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2}$ can be represented in the required form.

Define $C(n, m_1, m_2)$ as the set of all polynomials (33) such that $d_i = 0$, unless i is divisible either by m_1 or by m_2 , and $d_n \neq 0$. To be definite, we always will assume that n is divisible by m_1 . Similarly to the notation $C_i(Q)$ introduced above, for a polynomial $Q \in C(n, m_1, m_2)$ set

$$Ch_i(Q) = d_{n-i}, \quad 0 \leq i \leq n.$$

Lemma 3.4 *Let $Q \in C(n, m_1, m_2)$, where m_1 and m_2 are coprime, $m_1 \geq 3, m_2 \geq 3$.*

- 1) *If $Ch_1(Q) \neq 0$, then $Ch_2(Q) = 0$.*
- 2) *If $Ch_1(Q) \neq 0, Ch_3(Q) \neq 0$, then $m_1 = 3$, and $Ch_4(Q) = 0$,*
- 3) *If $Ch_1(Q) \neq 0, Ch_3(Q) \neq 0, Ch_5(Q) \neq 0$, then $m_1 = 3, m_2 = 4$.*

Proof. If $Ch_1(Q) \neq 0$, then $m_2 \mid n - 1$. Further, clearly

$$|m_i k_1 - m_i k_2| \geq m_i \geq 3, \quad i = 1, 2, \quad k_1, k_2 \in \mathbb{N}, \quad (34)$$

unless $k_1 = k_2$, implying that $n - 2$ may not be divisible neither by m_1 , since $m_1 \mid n$, nor by m_2 , since $m_2 \mid n - 1$. Thus, $Ch_2(Q) = 0$.

Assume that additionally $Ch_3(Q) \neq 0$. Since $m_2 \mid n - 1$, the number $n - 3$ may not be divisible by m_2 in view of (34). Therefore, $n - 3$ is divisible by m_1 implying that $m_1 = 3$. Furthermore, $Ch_4(Q) = 0$ for otherwise (34) implies that $m_2 = 3$ in contradiction with $\text{GCD}(m_1, m_2) = 1$.

Finally, if additionally $Ch_5(Q) \neq 0$, it follows from $m_1 = 3$ by (34) that $m_2 \mid n - 5$ implying that $m_2 = 4$. \square

Corollary 3.5 Let $Q \in C(n, m_1, m_2)$, where m_1 and m_2 are coprime, $m_1 \geq 3$, $m_2 \geq 3$, and n is even. If $Ch_1(Q) \neq 0$, then either $Ch_2(Q), Ch_3(Q)$ vanish, or $Ch_2(Q), Ch_4(Q), Ch_5(Q)$ vanish.

Proof. Since $m_2|n-1$, and n is even, m_2 is odd. Therefore, $m_2 \neq 4$, and the statement follows from Lemma 3.4. \square

Lemma 3.6 Let $Q = F \circ x^2 \circ (x - \delta)$, $\delta \in \mathbb{R}$, where $n = \deg Q \geq 6$.

1) If the coefficients $Ch_2(Q), Ch_3(Q)$ vanish, then either $\delta = 0$, or

$$(2\delta)^2 = \frac{3}{(n-1)(n-2)}. \quad (35)$$

2) If the coefficients $Ch_2(Q), Ch_4(Q), Ch_5(Q)$ vanish, then either $\delta = 0$, or 2δ satisfies the equation

$$\frac{2}{15}(n-1)(n-2)(n-3)(n-4)t^4 - (n-2)(n-3)t^2 + 1 = 0. \quad (36)$$

Proof. If $Ch_2(Q), Ch_3(Q)$ vanish, then $Q = c_0T_n + c_1T_{n-1} + R_1$, where $\deg R_1 \leq n-4$. Set

$$T_s^*(x) = 2T_s\left(\frac{x}{2}\right), \quad s \geq 1.$$

Clearly, the equality

$$c_0T_n + c_1T_{n-1} + R_1 = F \circ x^2 \circ (x - \delta)$$

implies the equality

$$c_0T_n^* + c_1T_{n-1}^* + \tilde{R}_1 = \tilde{F} \circ x^2 \circ (x - \gamma), \quad (37)$$

where $\tilde{R}_1 = 2R_1(x/2)$, $\tilde{F} = 2F(x/4)$, and $\gamma = 2\delta$. Furthermore, without loss of generality we may assume that $c_0 = 1$.

Since the right part of (37) is a polynomial in $(x - \gamma)^2$, it follows from (37) taking into account the inequality $\deg R_1 \leq n-4$ that the derivatives of $T_n^* + c_1T_{n-1}^*$ of orders $n-1$ and $n-3$ at the point γ vanish, that is

$$T_n^{*(n-1)}(\gamma) + c_1T_{n-1}^{*(n-1)}(\gamma) = 0,$$

$$T_n^{*(n-3)}(\gamma) + c_1T_{n-1}^{*(n-3)}(\gamma) = 0.$$

Since

$$T_s^* = x^s - sx^{s-2} + \frac{s(s-3)}{2}x^{s-4} - \frac{s(s-4)(s-5)}{6}x^{s-6} + \dots \quad (38)$$

by (12), this implies that

$$n!\gamma + c_1(n-1)! = 0,$$

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma + c_1\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) = 0.$$

The first of these equalities implies that $c_1 = -n\gamma$. Substituting this value of c_1 in the second equality we obtain

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma - n\gamma\left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)!\right) = -\frac{n!}{3}\gamma^3 + n(n-3)!\gamma = 0$$

implying that $2\delta = \gamma$ satisfies (35), unless $\delta = 0$.

Similarly, if $Ch_2(Q), Ch_4(Q), Ch_5(Q)$ vanish, we arrive to the equality

$$T_n^* + c_1T_{n-1}^* + c_3T_{n-3}^* + \tilde{R}_1 = \tilde{R} \circ x^2 \circ (x - \gamma), \quad (39)$$

where $\deg \tilde{R}_1 \leq n-6$ and $\gamma = 2\delta$, implying that the derivatives of $T_n^* + c_1T_{n-1}^* + c_3T_{n-3}^*$ of orders $n-1$ and $n-3$, and $n-5$ at the point γ vanish. Thus,

$$T_n^{*(n-1)}(\gamma) + c_1T_{n-1}^{*(n-1)}(\gamma) = 0,$$

$$T_n^{*(n-3)}(\gamma) + c_1 T_{n-1}^{*(n-3)}(\gamma) + c_3 T_{n-3}^{*(n-3)}(\gamma) = 0,$$

$$T_n^{*(n-5)}(\gamma) + c_1 T_{n-1}^{*(n-5)}(\gamma) + c_3 T_{n-3}^{*(n-5)}(\gamma) = 0,$$

or equivalently

$$n!\gamma + c_1(n-1)! = 0,$$

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma + c_1 \left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)! \right) + c_3(n-3)! = 0,$$

and

$$\begin{aligned} & \frac{n!}{5!}\gamma^5 - \frac{n(n-2)!}{3!}\gamma^3 + \frac{n(n-3)(n-4)!}{2}\gamma + \\ & + c_1 \left(\frac{(n-1)!}{4!}\gamma^4 - \frac{(n-1)(n-3)!}{2!}\gamma^2 + \frac{(n-1)(n-4)(n-5)!}{2} \right) + \\ & + c_3 \left(\frac{(n-3)!}{2!}\gamma^2 - (n-3)(n-5)! \right) = 0. \end{aligned}$$

As above, it follows from the first of these equalities by (38) that $c_1 = -n\gamma$ and substituting this value of c_1 in the second equality we obtain

$$\frac{n!}{3!}\gamma^3 - n(n-2)!\gamma - n\gamma \left(\frac{(n-1)!}{2!}\gamma^2 - (n-1)(n-3)! \right) + c_3(n-3)! = 0$$

implying that

$$c_3 = \frac{n(n-1)(n-2)}{3}\gamma^3 - n\gamma.$$

Now the third equality gives us

$$\begin{aligned} & \frac{n!}{5!}\gamma^5 - \frac{n(n-2)!}{3!}\gamma^3 + \frac{n(n-3)(n-4)!}{2}\gamma - n\gamma \left(\frac{(n-1)!}{4!}\gamma^4 - \frac{(n-1)(n-3)!}{2!}\gamma^2 + \frac{(n-1)(n-4)(n-5)!}{2} \right) + \\ & + \left(\frac{n(n-1)(n-2)}{3}\gamma^3 - n\gamma \right) \left(\frac{(n-3)!}{2!}\gamma^2 - (n-3)(n-5)! \right) = 0. \end{aligned}$$

The coefficient of γ^5 in this expression is

$$\frac{n!}{3!} \left(\frac{1}{20} - \frac{1}{4} + 1 \right) = \frac{2n!}{15}.$$

The coefficient of γ^3 is

$$\begin{aligned} & -\frac{n(n-2)!}{3!} + \frac{n(n-1)(n-3)!}{2!} - \frac{n(n-1)(n-2)(n-3)(n-5)!}{3} - \frac{n(n-3)!}{2!} = \\ & = -\frac{n(n-3)!}{2!} \left(\frac{n-2}{3} - (n-1) \right) - n(n-3)(n-5)! \left(\frac{(n-1)(n-2)}{3} + \frac{(n-4)}{2} \right) = \\ & = \frac{(2n-1)n(n-3)!}{6} - n(n-3)(n-5)! \left(\frac{2n^2-3n-8}{6} \right) = \\ & = \frac{n(n-3)(n-5)!}{6} ((2n-1)(n-4) - (2n^2-3n-8)) = -n(n-2)(n-3)(n-5)! \end{aligned}$$

Finally, the coefficient of γ is

$$\begin{aligned} & \frac{n(n-3)(n-4)!}{2} - \frac{n(n-1)(n-4)(n-5)!}{2} + n(n-3)(n-5)! = \\ & = n(n-5)! \left(\frac{(n-3)(n-4)}{2} - \frac{(n-1)(n-4)}{2} + (n-3) \right) = n(n-5)!. \end{aligned}$$

Collecting terms and canceling by $n(n-5)!$ we see that $2\delta = \gamma$ satisfies (36), unless $\delta = 0$. \square

Corollary 3.7 *Let $Q = R \circ x^2 \circ (x - \delta)$, $\delta \in \mathbb{R}$.*

1) If the coefficients $Ch_2(Q)$, $Ch_3(Q)$ vanish and $n = \deg Q \geq 6$, then the number 4δ is not an algebraic integer, unless $\delta = 0$.

2) If the coefficients $Ch_2(Q)$, $Ch_4(Q)$, $Ch_5(Q)$ vanish and $n = \deg Q \geq 9$, then the number 4δ is not an algebraic integer, unless $\delta = 0$.

Proof. Set $\gamma = 4\delta$. If $Ch_2(Q)$, $Ch_3(Q)$ vanish and $\delta \neq 0$, then γ is a root of the equation

$$t^2 - \frac{12}{(n-1)(n-2)} = 0.$$

Since for $n \geq 6$ the number $\frac{12}{(n-1)(n-2)}$ is not an integer, this implies that γ cannot be an algebraic integer of degree two. Moreover, γ cannot be an algebraic integer of degree one for otherwise γ is an integer implying that $\gamma^2 = \frac{12}{(n-1)(n-2)}$ is also an integer.

If $Ch_2(Q)$, $Ch_4(Q)$, $Ch_5(Q)$ vanish and $\delta \neq 0$, then γ is a root of the equation

$$(n-1)(n-2)(n-3)(n-4)t^4 - 30(n-2)(n-3)t^2 + 120 = 0. \quad (40)$$

If this equation is irreducible over \mathbb{Q} , then γ cannot be an algebraic integer since the number $(n-1)(n-2)(n-3)(n-4)$ obviously does not divide 120 for $n \geq 9$.

Assume now that γ is an algebraic integer satisfying an irreducible equation $t^2 + c_1t + c_2 = 0$, $c_1, c_2 \in \mathbb{Z}$. Then by the Gauss lemma the equality

$$(n-1)(n-2)(n-3)(n-4)t^4 - 30(n-2)(n-3)t^2 + 120 = (t^2 + c_1t + c_2)(d_0t^2 + d_1t + d_2)$$

holds for some $d_0, d_1, d_2 \in \mathbb{Z}$. Since the coefficients of t^3 and t in the left part vanish we have:

$$d_1 + c_1d_0 = 0, \quad c_1d_2 + c_2d_1 = 0,$$

implying that

$$d_1 = -c_1d_0, \quad d_2 = c_2d_0, \quad (41)$$

unless

$$c_1 = 0, \quad d_1 = 0. \quad (42)$$

If (41) holds, then $120 = c_2d_2 = c_2^2d_0$ in contradiction with $d_0 = (n-1)(n-2)(n-3)(n-4)$ and $n \geq 9$. Similarly, if (42) holds, then $\gamma^2 = -c_2$ is an integer, and it follows from (40) that the number $(n-2)(n-3)\gamma^2$ divides 120 implying easily a contradiction with the condition $n \geq 9$.

Finally, observe that if γ is a rational root of (40), then $-\gamma$ also is a root of (40) and $t^2 - \gamma^2$ is a divisor of (40). Therefore, any irreducible over \mathbb{Q} factor of (40) has the degree one, two, or four. Thus, in order to finish the proof it is enough to observe that γ is not an integer, since otherwise γ^2 is also an integer implying as above that $(n-2)(n-3)\gamma^2$ divides 120. \square

Proof of Theorem 1.2 in the case (2,1). Observe first that if $\beta = 0$ in (31), then the theorem is true. Indeed, in this case the condition $\widetilde{W}_1(a) = \widetilde{W}_1(b)$ is equivalent to the condition $T_2(a) = T_2(b)$. Therefore, applying Lemma 2.2, b) to the equalities

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b) \quad T_2(a) = T_2(b),$$

we conclude that at least one of the numbers m_1, m_2 is even and hence $Q = \widetilde{U} \circ xR^2(x) \circ (\alpha x)^2$ and $P = U \circ T_{m_1m_2} = U \circ T_{m_1m_2/2} \circ T_2$ satisfy (4) for $W = x^2$.

Further, observe that (31) implies that

$$Q = \widetilde{U} \circ xR^2(x) \circ x^2 \circ (\alpha x + \beta) = F \circ x^2 \circ \left(x + \frac{\beta}{\alpha}\right), \quad (43)$$

where $F = \widetilde{U} \circ xR^2(x) \circ \alpha^2 x$, while the condition $\widetilde{W}_1(a) = \widetilde{W}_1(b)$ yields that

$$a + b = -\frac{2\beta}{\alpha}. \quad (44)$$

If $Ch_1(Q) = 0$, then (33) implies that also $C_1(Q) = 0$, since $C_1(T_n) = 0$ by (12). In its turn $C_1(Q) = 0$ implies that $\beta = 0$ by Corollary 3.2 applied to (43). So, assume that $Ch_1(Q) \neq 0$. By Corollary 3.5, this implies that either $Ch_2(Q), Ch_3(Q)$ vanish, or $Ch_2(Q), Ch_4(Q), Ch_5(Q)$ vanish.

If $Ch_2(Q), Ch_3(Q)$ vanish, then Corollary 3.7, 1) applied to (43) implies that the number $-4\beta/\alpha$ is not an algebraic integer, unless $\beta = 0$. On the other hand, (44) implies that $-4\beta/\alpha$ is an algebraic integer, since $2a$ and $2b$ are algebraic integers by Corollary 2.5. Thus, we conclude again that $\beta = 0$.

Similarly, assuming that $Ch_3(Q) \neq 0$, while $Ch_2(Q), Ch_4(Q), Ch_5(Q)$ vanish, we may apply Corollary 3.7, 2), whenever $n \geq 9$. Therefore, since $Ch_3(Q) \neq 0$ implies that $3|n$ in view of the equality $m_1 = 3$, the only case which remains uncovered is the one where $n = 6$. In this case the inequality $Ch_1(Q) \neq 0$ implies that $m_2 = 5$. Notice that it follows from (40) that for $n = 6$ the number $4\beta/\alpha$ satisfies the equation $t^4 - 3t^2 + 1 = 0$ whose roots are algebraic integers. Moreover, the system

$$T_5(a) = T_5(b), \quad T_3(a) = T_3(b), \quad (2a + 2b)^4 - 3(2a + 2b)^2 + 1 = 0$$

has non-zero solutions. Thus, for $n = 6$ the previous reasoning fails.

In order to prove the theorem in this case remind that in Lemma 3.6 we used a condition which is weaker than the one in (31). Therefore, in order to finish the proof it is enough to show that the equality

$$T_6 + c_1 T_5 + c_3 T_3 + c_6 = c(z(z^2 - d))^2 \circ (z - \beta), \quad (45)$$

where $c_1, c_3, c_6, c, d, \beta \in \mathbb{R}$, is possible only if $c_1 = 0$. This statement may be verified by a direct calculation. Namely, the comparison of leading coefficients of both parts of (45) implies that $c = 32$, while the comparison of other coefficients gives

$$\begin{aligned} 16c_1 + 192b &= 0, & -480b^2 + 64d - 48 &= 0, \\ 640b^3 - 256bd - 20c_1 + 4c_3 &= 0, & -480b^4 + 384b^2d - 32d^2 + 18 &= 0, \\ 192b^5 - 256b^3d + 64bd^2 + 5c_1 - 3c_3, & & c_6 &= -32b^6 + 64b^4d - 32b^2d^2 - 1. \end{aligned}$$

We leave the reader to verify (for example, with the help of Maple) that the only solution of the above system is

$$c_1 = 0, \quad c_3 = 0, \quad c_6 = 1, \quad b = 0, \quad d = 3/4,$$

(for these values of parameters equality (45) simply reduces to the equality $T_6 = T_2 \circ T_3$).

3.4 Proof of Theorem 1.2 in the case (2,2)

First, observe that Theorem 1.2 in the case (2,2) follows from the following statement.

Proposition 3.8 *Let $V_1, V_2, U, \tilde{V}_1, \tilde{V}_2, \tilde{U} \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, satisfy the equalities*

$$Q = V_1 \circ T_{m_1} + V_2 \circ T_{m_2} = \tilde{U} \circ T_{\tilde{m}_1 \tilde{m}_2} \circ (\alpha x + \beta), \quad (46)$$

and

$$P = \tilde{V}_1 \circ T_{\tilde{m}_1} \circ (\alpha x + \beta) + \tilde{V}_2 \circ T_{\tilde{m}_2} \circ (\alpha x + \beta) = U \circ T_{m_1 m_2}, \quad (47)$$

where

$$\text{GCD}(m_1, m_2) = 1, \quad \text{GCD}(\tilde{m}_1, \tilde{m}_2) = 1, \quad (48)$$

and $m_1 \geq 3, m_2 \geq 3, \tilde{m}_1 \geq 3, \tilde{m}_2 \geq 3$. Then $\alpha = \pm 1, \beta = 0$.

Indeed, in the case (2,2) conditions (46), (47), (48) are satisfied for some $m_1 \geq 2, m_2 \geq 2, \tilde{m}_1 \geq 2, \tilde{m}_2 \geq 2$. Additionally, the equalities

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b), \quad (49)$$

$$T_{\tilde{m}_1}(\alpha a + \beta) = T_{\tilde{m}_1}(\alpha b + \beta), \quad T_{\tilde{m}_2}(\alpha a + \beta) = T_{\tilde{m}_2}(\alpha b + \beta) \quad (50)$$

hold. If at least one of the numbers m_1, m_2 equals 2, then P, Q belongs to the type (1,2) considered above. So, we may assume that $m_1 \geq 3, m_2 \geq 3$. Similarly, we may assume that $\tilde{m}_1 \geq 3, \tilde{m}_2 \geq 3$, since otherwise P, Q belongs to the type (2,1). Since under these conditions Proposition 3.8 implies that $\alpha = \pm 1, \beta = 0$, it follows from the equalities

$$T_{m_1}(a) = T_{m_1}(b), \quad T_{m_2}(a) = T_{m_2}(b), \quad T_{\tilde{m}_1}(\pm a) = T_{\tilde{m}_1}(\pm b)$$

by Lemma 2.2, b), taking into account (13), that $T_s(a) = T_s(b)$, where either $s = \text{GCD}(m_1, \tilde{m}_1)$, or $s = \text{GCD}(m_2, \tilde{m}_2)$. Since

$$Q = \tilde{U} \circ T_{\tilde{m}_1 \tilde{m}_2} \circ (\pm z) = \tilde{U} \circ (\pm T_{\tilde{m}_1 \tilde{m}_2}) = \tilde{U} \circ (\pm T_{\tilde{m}_1 \tilde{m}_2/s}) \circ T_s$$

and

$$P = U \circ T_{m_1 m_2} = U \circ T_{m_1 m_2/s} \circ T_s,$$

we conclude that (4) holds for $W = T_s$.

The next lemmas are similar to Lemma 3.4 and Lemma 3.6 and are used for imposing restrictions on possible values of α and β in (46), (47), and eventually to show that $\alpha = \pm 1, \beta = 0$.

Lemma 3.9 *Let $Q = F \circ T_s \circ (\alpha x + \beta)$, where $s \geq 5, \alpha, \beta \in \mathbb{R}, \alpha \neq 0$.*

1) *If the coefficients $Ch_2(Q), Ch_3(Q)$ vanish, then either $\alpha = \pm 1, \beta = 0$, or*

$$4\beta^2 = \frac{6}{(n-1)(2n-1)}, \quad \alpha^2 = \frac{2n-4}{2n-1}. \quad (51)$$

2) *If the coefficients $Ch_2(Q), Ch_4(Q)$, vanish, then either $\alpha = \pm 1, \beta = 0$, or*

$$4\beta^2 = \frac{12}{(n-1)(2n-1)}, \quad \alpha^2 = \frac{2n-7}{2n-1}. \quad (52)$$

In particular, in both cases $\alpha^2 < 1$ and $\beta \neq 0$, unless $\alpha = \pm 1, \beta = 0$.

Proof. Set $n = \deg Q$. By Lemma 3.1, for some $b_0 \in \mathbb{R}$ we have:

$$C_i(Q) = C_i(F \circ T_s \circ (\alpha x + \beta)) = C_i(b_0 T_{n/s} \circ T_s \circ (\alpha x + \beta)) = C_i(b_0 T_n \circ (\alpha x + \beta)), \quad 0 \leq i \leq s-1,$$

implying that

$$Q = (b_0 T_n) \circ (\alpha x + \beta) + R_2,$$

where R_2 is a polynomial such that $\deg R_2 \leq n - s$. Thus, if $Ch_2(Q) = 0$, we have:

$$Q = c_0 T_n + c_1 T_{n-1} + c_3 T_{n-3} + c_4 T_{n-4} + R_1 = b_0 T_n \circ (\alpha x + \beta) + R_2, \quad (53)$$

where $\deg R_1, \deg R_2 \leq n - 5$, and $c_0, c_1, c_3, c_4, b_0 \in \mathbb{R}$. Changing x to $x/2$, and $R_i(x)$ to $2R_i(x/2)$, we obtain a similar equality

$$Q = c_0 T_n^* + c_1 T_{n-1}^* + c_3 T_{n-3}^* + c_4 T_{n-4}^* + R_1 = b_0 T_n^* \circ (\tilde{\alpha} x + \tilde{\beta}) + R_2, \quad (54)$$

where $\tilde{\beta} = 2\beta, \tilde{\alpha} = \alpha$. Furthermore, without loss of generality we may assume that $c_0 = 1$, implying that $b_0 = 1/\tilde{\alpha}^n$ and $c_1 = \tilde{\beta}n/\tilde{\alpha}$. Thus, we can rewrite (54) in the form

$$Q = T_n^* + \frac{\tilde{\beta}n}{\tilde{\alpha}} T_{n-1}^* + c_3 T_{n-3}^* + c_4 T_{n-4}^* + R_1 = \frac{1}{\tilde{\alpha}^n} T_n^* \circ (\tilde{\alpha} x + \tilde{\beta}) + R_2. \quad (55)$$

Calculating $C_2(Q), C_3(Q), C_4(Q)$ for both representations of Q in (55) using formula (38) (and the Taylor formula, for the second representation), we obtain the equalities

$$-n = \frac{1}{(n-2)!\tilde{\alpha}^2} \left[\frac{n!\tilde{\beta}^2}{2!} - n(n-2)! \right],$$

$$\begin{aligned} \frac{-\tilde{\beta}n(n-1)}{\tilde{\alpha}} + c_3 &= \frac{1}{(n-3)!\tilde{\alpha}^3} \left[\frac{n!\tilde{\beta}^3}{3!} - n(n-2)!\tilde{\beta} \right], \\ \frac{n(n-3)}{2} + c_4 &= \frac{1}{(n-4)!\tilde{\alpha}^4} \left[\frac{n!\tilde{\beta}^4}{4!} - \frac{n(n-2)!\tilde{\beta}^2}{2!} + \frac{n(n-3)(n-4)!}{2} \right]. \end{aligned}$$

It follows from the first of these equalities that

$$\tilde{\alpha}^2 = 1 - \frac{(n-1)\tilde{\beta}^2}{2}. \quad (56)$$

If $c_3 = 0$, then substituting this value of $\tilde{\alpha}^2$ into third equation we obtain that either $\beta = 0$ and then $\alpha = \pm 1$ by (56), or (51) holds. Similarly, if $c_4 = 0$, then substituting the value of $\tilde{\alpha}^2$ from (56) into the fourth equation we obtain

$$\frac{n!\tilde{\beta}^4}{4!} - \frac{n(n-2)!\tilde{\beta}^2}{2!} + \frac{n(n-3)(n-4)!}{2} = \frac{(n-4)!n(n-3)}{2} \left[\frac{(n-1)^2\tilde{\beta}^4}{4} - (n-1)\tilde{\beta}^2 + 1 \right]$$

implying that, either $\tilde{\alpha} = \pm 1$, $\tilde{\beta} = 0$, or (52) holds.

Finally, clearly $\alpha^2 < 1$, $\beta \neq 0$, unless $\alpha = \pm 1$, $\beta = 0$. \square

Lemma 3.10 *Let $Q \in C(n, m_1, m_2)$, where m_1 and m_2 are coprime, $m_1 \geq 3$, $m_2 \geq 3$. Then at least one of the coefficients $Ch_2(Q)$, $Ch_4(Q)$, $Ch_6(Q)$ vanishes.*

Proof. Assume that $Ch_2(Q) \neq 0$, $Ch_4(Q) \neq 0$, and show that this implies that $Ch_6(Q) = 0$. First, $Ch_2(Q) \neq 0$ implies by (34) that $m_2 \mid n-2$. It follows now from $Ch_4(Q) \neq 0$ by (34) that $m_1 = 4$. Therefore, $Ch_6(Q) = 0$ for otherwise (34) implies that $m_2 = 4$ in contradiction with $\text{GCD}(m_1, m_2) = 1$. \square

Lemma 3.11 *Let $Q = U \circ T_s \circ \alpha x$, where $U \in \mathbb{R}[x]$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $s \geq 6$.*

- 1) *If the coefficient $Ch_2(Q)$ vanishes, then $\alpha^2 = 1$.*
- 2) *If the coefficient $Ch_4(Q)$ vanishes, then either $\alpha^2 = 1$, or $\alpha^2 = \frac{n-3}{n-1}$.*
- 3) *If the coefficient $Ch_6(Q)$ vanishes, then α^2 is a root of the equation*

$$(n^2 - 3n + 2)t^2 + (-2n^2 + 12n - 16)t + (n^2 - 9n + 20) = 0. \quad (57)$$

In particular, in all the cases the inequality $\alpha^2 < 1$ holds, unless $\alpha^2 = 1$.

Proof. Set $n = \deg Q$. As in Lemma 3.9 we arrive to the equality

$$T_n^* + c_2 T_{n-2}^* + c_4 T_{n-4}^* + c_6 T_{n-6}^* + R_1 = \frac{1}{\tilde{\alpha}^n} T_n^* \circ (\alpha x) + R_2, \quad (58)$$

where $\deg R_1, \deg R_2 \leq n-7$ (since obviously $c_1 = c_3 = c_5 = 0$). Calculating now $C_2(Q)$, $C_4(Q)$, $C_6(Q)$ for both representations of Q in (58), we obtain:

$$\begin{aligned} -n + c_2 &= -\frac{n}{\alpha^2}, \\ \frac{n(n-3)}{2} - (n-2)c_2 + c_4 &= \frac{n(n-3)}{2\alpha^4}, \\ -\frac{n(n-4)(n-5)}{6} + \frac{(n-2)(n-5)c_2}{2} - (n-4)c_4 + c_6 &= -\frac{n(n-4)(n-5)}{6\alpha^6} \end{aligned}$$

It follows from the first equality that

$$c_2 = \frac{n(\alpha^2 - 1)}{\alpha^2},$$

implying that if $c_2 = 0$, then $\alpha^2 = 1$. Substituting now the value of c_2 into the second equality we obtain that

$$c_4 = \frac{n(n-3)}{2\alpha^4} - \frac{n(n-3)}{2} + \frac{n(n-2)(\alpha^2-1)}{\alpha^2} = n \left(\frac{(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3)}{2\alpha^4} \right).$$

Since

$$(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3) = (n-1)(\alpha^2-1) \left(\alpha^2 - \frac{n-3}{n-1} \right),$$

this implies that if $c_4 = 0$, then either $\alpha^2 = 1$, or $\alpha^2 = \frac{n-3}{n-1}$.

Finally,

$$\begin{aligned} c_6 &= -\frac{n(n-4)(n-5)}{6\alpha^6} + \frac{n(n-4)(n-5)}{6} - \frac{(n-2)(n-5)c_2}{2} + (n-4)c_4 = \\ &= -\frac{n(n-4)(n-5)}{6\alpha^6} + \frac{n(n-4)(n-5)}{6} - \frac{(n-2)(n-5)}{2} \frac{n(\alpha^2-1)}{\alpha^2} + \\ &\quad + (n-4)n \left(\frac{(n-1)\alpha^4 - 2(n-2)\alpha^2 + (n-3)}{2\alpha^4} \right) = \\ &= n \left(\frac{(n^2-3n+2)\alpha^6 + (-3n^2+15n-18)\alpha^4 + (3n^2-21n+36)\alpha^2 - n^2 + 9n - 20}{6\alpha^6} \right). \end{aligned}$$

Now the statement follows from the factorization

$$\begin{aligned} (n^2-3n+2)\alpha^6 + (-3n^2+15n-18)\alpha^4 + (3n^2-21n+36)\alpha^2 - n^2 + 9n - 20 &= \\ &= (\alpha^2-1) \left((n^2-3n+2)\alpha^4 + (-2n^2+12n-16)\alpha^2 + (n^2-9n+20) \right). \end{aligned}$$

In order to finish the proof we only must show that the absolute values of roots of equation (57) are less than one. Solving (57), we find two roots

$$\begin{aligned} t_1 &= \frac{n^2-6n+8-\sqrt{3n^2-18n+24}}{n^2-3n+2} = \frac{\sqrt{(n-2)(n-4)}(\sqrt{(n-2)(n-4)}-\sqrt{3})}{(n-1)(n-2)}, \\ t_2 &= \frac{n^2-6n+8+\sqrt{3n^2-18n+24}}{n^2-3n+2} = \frac{\sqrt{(n-2)(n-4)}(\sqrt{(n-2)(n-4)}+\sqrt{3})}{(n-1)(n-2)}. \end{aligned}$$

Clearly, for $n \geq 6$ the inequality $0 < t_1 < t_2$ holds. Finally,

$$t_2 - 1 = \frac{\sqrt{3(n-2)(n-4)} - 3n + 6}{(n-1)(n-2)} = \frac{\sqrt{3(n-2)}(\sqrt{n-4} - \sqrt{3(n-2)})}{(n-1)(n-2)} < 0. \quad \square$$

Proof of Proposition 3.8. Assume first that $Ch_1(Q) \neq 0$. Then Lemma 3.4 implies that either $Ch_2(Q)$, $Ch_3(Q)$ vanish, or $Ch_2(Q)$, $Ch_4(Q)$ vanish. It follows now from Lemma 3.9 that, unless $\alpha = \pm 1$, $\beta = 0$, the conditions $\alpha < 1$, $\beta \neq 0$ hold. So, assume that $\alpha < 1$, $\beta \neq 0$.

Rewrite equality (47) in the form

$$P = \tilde{V}_1 \circ T_{\tilde{m}_1} + \tilde{V}_2 \circ T_{\tilde{m}_2} = U \circ T_{m_1 m_2} \circ \left(\frac{x-\beta}{\alpha} \right). \quad (59)$$

Since $\beta \neq 0$, Corollary 3.2 applied to (59) implies that $C_1(P) \neq 0$. Applying now Lemma 3.4 and Lemma 3.9 to (59) in the same way as before to equality (46), we conclude that $1/\alpha < 1$. The contradiction obtained proves that $\alpha = \pm 1$, $\beta = 0$.

Assume now that $Ch_1(Q) = 0$. Then $\beta = 0$, by Corollary 3.2. Furthermore, by Lemma 3.10 at least one of the coefficients $Ch_2(Q)$, $Ch_4(Q)$, $Ch_6(Q)$ vanish, implying by Lemma 3.11 that, unless $\alpha = \pm 1$, $\beta = 0$, the condition $\alpha < 1$ holds. Since $\beta = 0$ implies by Corollary 3.2 that $C_1(P) = 0$ in view of (59), the assumption $\alpha < 1$ leads to a contradiction in the same way as before. \square

3.5 Proof of Theorem 1.2 in the cases (3,1), (3,2), (3,3).

The case (3,1) reduces to the case (2,1) as follows. We start from the equality

$$Q = V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} + V_3 \circ xR(x^2) \circ T_{m_1m_2} = \tilde{U} \circ x^2\tilde{R}^2(x^2) \circ (\alpha x + \beta), \quad (60)$$

where $V_1, V_2, V_3, R, \tilde{R}, \tilde{U} \in \mathbb{R}[x]$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, and $m_1 \geq 3$, $m_2 \geq 3$ are coprime and odd. It follows from the first representation for Q in (60) that Q can be written in the form

$$Q = d_n T_n + d_{n-1} T_{n-1} + \cdots + d_1 T_1 + d_0, \quad d_i \in \mathbb{R}, \quad (61)$$

where $d_i = 0$, unless i is divisible either by $2m_1$, or by $2m_2$, or by m_1m_2 . Clearly, conditions imposed on m_1, m_2 imply that

$$|2m_1k_1 - 2m_2k_2| \geq 2, \quad \text{unless} \quad |2m_1k_1 - 2m_2k_2| = 0, \quad (62)$$

$$|2m_ik_1 - 2m_ik_2| \geq 2m_i \geq 6, \quad \text{unless} \quad |2m_ik_1 - 2m_ik_2| = 0, \quad i = 1, 2 \quad (63)$$

$$|2m_ik_1 - m_1m_2k_2| \geq m_i \geq 3, \quad \text{unless} \quad |2m_ik_1 - m_1m_2k_2| = 0, \quad i = 1, 2 \quad (64)$$

$$|m_1m_2k_1 - m_1m_2k_2| \geq m_1m_2 \geq 15, \quad \text{unless} \quad |m_1m_2k_1 - m_1m_2k_2| = 0. \quad (65)$$

Therefore, $Ch_1(Q) = 0$, implying that $C_1(Q) = 0$, since $C_1(T_n) = 0$. It follows now from the second representation for Q in (60) by Corollary 3.2 that $\beta = 0$. Since the polynomial $\tilde{W}_1 = z^2 \circ (\alpha x + \beta)$ satisfies $\tilde{W}_1(a) = \tilde{W}_1(b)$, this implies that $a = -b$. Therefore, solution P, Q also belongs to the case (2,1) considered earlier (see the remarks after Theorem 2.1).

In the case (3,2) there exist $V_1, V_2, V_3, U, R, \tilde{V}_1, \tilde{V}_2, \tilde{U} \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$Q = V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} + V_3 \circ xR(x^2) \circ T_{m_1m_2} = \tilde{U} \circ T_{\tilde{m}_1\tilde{m}_2} \circ (\alpha x + \beta), \quad (66)$$

$$P = \tilde{V}_1 \circ T_{\tilde{m}_1} \circ (\alpha x + \beta) + \tilde{V}_2 \circ T_{\tilde{m}_2} \circ (\alpha x + \beta) = U \circ x^2R^2(x^2) \circ T_{m_1m_2} \quad (67)$$

where $m_1 \geq 3$, $m_2 \geq 3$ are odd, $\text{GCD}(m_1, m_2) = 1$, and $\tilde{m}_1 \geq 2$, $\tilde{m}_2 \geq 2$, $\text{GCD}(\tilde{m}_1, \tilde{m}_2) = 1$. Further, without loss of generality we may assume that $a \neq -b$, for otherwise P, Q belongs to the case (2,2). Besides, we may assume that $\tilde{m}_1 \geq 3$, $\tilde{m}_2 \geq 3$, for otherwise P, Q belongs to the case (3,1).

Since equalities (66), (67) may be written in the form

$$Q = (V_1 \circ T_2 + V_3 \circ xR(x^2) \circ T_{m_2}) \circ T_{m_1} + (V_2 \circ T_2) \circ T_{m_2} = \tilde{U} \circ T_{\tilde{m}_1\tilde{m}_2} \circ (\alpha x + \beta),$$

$$P = \tilde{V}_1 \circ T_{\tilde{m}_1} \circ (\alpha x + \beta) + \tilde{V}_2 \circ T_{\tilde{m}_2} \circ (\alpha x + \beta) = (U \circ x^2R^2(x^2)) \circ T_{m_1m_2},$$

it follows from Proposition 3.8 that $\alpha = \pm 1$, $\beta = 0$. Since we assumed that $a \neq -b$, it follows now from the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \quad T_{2m_2}(a) = T_{2m_2}(b), \quad T_{\tilde{m}_1}(\pm a) = T_{\tilde{m}_1}(\pm b)$$

by Lemma 2.2, b), taking into account (13), that $T_s(a) = T_s(b)$, where either $s = \text{GCD}(2m_1, \tilde{m}_1)$, or $s = \text{GCD}(2m_2, \tilde{m}_2)$. Finally, since

$$Q = \tilde{U} \circ T_{\tilde{m}\tilde{n}} \circ (\pm z) = \tilde{U} \circ (\pm T_{\tilde{m}\tilde{n}}) = \tilde{U} \circ (\pm T_{\tilde{m}\tilde{n}/s}) \circ T_s$$

and

$$P = U \circ x^2R^2(x^2) \circ T_{m_1m_2} = U \circ x^2R^2(x^2) \circ T_{m_1m_2/s} \circ T_s,$$

we conclude that (4) holds for $W = T_s$.

The proof in the case (3,3) is similar: there exist $V_1, V_2, V_3, U, R, \tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{U}, \tilde{R} \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$Q = V_1 \circ T_{2m_1} + V_2 \circ T_{2m_2} + V_3 \circ xR(x^2) \circ T_{m_1m_2} = \tilde{U} \circ x^2\tilde{R}^2(x^2) \circ T_{\tilde{m}_1\tilde{m}_2} \circ (\alpha x + \beta),$$

$$P = \tilde{V}_1 \circ T_{\tilde{2m}_1} \circ (\alpha x + \beta) + \tilde{V}_2 \circ T_{\tilde{2m}_2} \circ (\alpha x + \beta) + \tilde{V}_3 \circ x\tilde{R}(x^2) \circ T_{\tilde{m}_1\tilde{m}_2} = U \circ x^2R^2(x^2) \circ T_{m_1m_2}$$

where $m_1 \geq 3$, $m_2 \geq 3$, $\tilde{m}_1 \geq 3$, $\tilde{m}_2 \geq 3$ are odd, $\text{GCD}(m_1, m_2) = 1$, $\text{GCD}(\tilde{m}_1, \tilde{m}_2) = 1$. Moreover, without loss of generality we may assume that $a \neq -b$. Further, using Proposition 3.8 we conclude as above that $\alpha = \pm 1$, $\beta = 0$. Finally, it follows from the equalities

$$T_{2m_1}(a) = T_{2m_1}(b), \quad T_{2m_2}(a) = T_{2m_2}(b), \quad T_{\tilde{2m}_1}(\pm a) = T_{\tilde{2m}_1}(\pm b)$$

that (4) holds for $W = T_s$, where either $s = \text{GCD}(2m_1, 2\tilde{m}_1)$, or $s = \text{GCD}(2m_2, 2\tilde{m}_2)$.

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